Quantum Mechanics A
Fall 2010
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Problem Set 1

Problem 1. Consider a general operator $\hat{M}$ which in the x, y photon-polarization basis is the following 2×2 matrix

$$\hat{M} = \begin{pmatrix} m_{xx} & m_{xy} \\ m_{yx} & m_{yy} \end{pmatrix}, \tag{1}$$

$$m_{ij} = \langle i|\hat{M}|j\rangle \tag{2}$$

where $|i\rangle$ or $|j\rangle$ stand for any of the two basis vectors $|x\rangle$ and $|y\rangle$ corresponding to the two photon polarizations.

1. Show that

$$\langle\Phi|\hat{M}|\Psi\rangle^* = \langle\Psi|\hat{M}^\dagger|\Phi\rangle \tag{3}$$

for any two states $|\Phi\rangle$ and $|\Psi\rangle$ which, in general, are linear combinations of $|x\rangle$ and $|y\rangle$, i.e,

$$|\Phi\rangle = \Phi_x|x\rangle + \Phi_y|y\rangle, \tag{4}$$

$$|\Psi\rangle = \Psi_x|x\rangle + \Psi_y|y\rangle. \tag{5}$$

2. Show that if $|\Psi\rangle$ is chosen such as

$$\hat{M}|\Psi\rangle = \lambda|\Psi\rangle \tag{6}$$

then, necessarily the following follows from that

$$\langle\Psi|\hat{M}^\dagger = \lambda^*\langle\Psi| \tag{7}$$

3. Show that the transformation from basis to another is a unitary transformation.

4. Show that the matrix $|\Phi\rangle\langle\Phi|$ is Hermitian.

5. Show that the photon spin operator $\hat{S}$ discussed in class is Hermitian. Generally all physical quantities are represented by Hermitian operators.

Problem 2. Let $|x(\theta)\rangle$ and $|y(\theta)\rangle$ denote the basis rotated by an angle to the $|x\rangle$, $|y\rangle$ basis, which correspond to the two distinct photon polarizations. Show that the two components of a state vector $|\Psi\rangle$ in this rotated basis, i.e., $\langle x(\theta)|\Psi\rangle$ and $\langle y(\theta)|\Psi\rangle$ obey the following differential equations,

$$-i\frac{\partial}{\partial\theta}\langle x(\theta)|\Psi\rangle = \langle x(\theta)|\hat{S}|\Psi\rangle, \tag{8}$$

$$-i\frac{\partial}{\partial\theta}\langle y(\theta)|\Psi\rangle = \langle y(\theta)|\hat{S}|\Psi\rangle, \tag{9}$$

Problem 3 The trace of an operator $\hat{O}$ in an n dimensional linear space is the sum of all the diagonal elements of the $n \times n$ matrix O_{nm} which represents the operator $\hat{O}$ in a given basis.

Show that the trace of any such matrix representing an operator is independent of the choice of the basis.

Hint: Use the fact that a transformation from one basis to another is a unitary transformation (part 3 of Problem 1).

Problem 4. Suppose that a photon is in a state $|\Psi\rangle$. Let

$$\hat{P}_\Psi = |\Psi\rangle\langle\Psi|. \tag{10}$$

1. Show that the expectation value for the photon of a physical quantity represented by the operator $\hat{Q}$ is

$$\langle\Psi|\hat{Q}|\Psi\rangle = tr(\hat{P}_\Psi\hat{Q}), \tag{11}$$

   where tr stands for the trace defined in the previous problem.

2. Suppose now that the photon is in a state that is a mixture of the state $|\Psi_1\rangle$ with probability p_1, $|\Psi_2\rangle$ with probability p_2, etc., where

$$\sum_i p_i = 1. \tag{12}$$

Let,

$$\hat{\rho} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|. \tag{13}$$

Show that the expectation value for this mixed state of a physical quantity represented by the operator $\hat{Q}$ is

$$\langle\hat{Q}\rangle = tr(\hat{\rho}\hat{Q}), \tag{14}$$

3. The matrix $\hat{\rho}$ is called the *density matrix*. Show that

$$tr(\hat{\rho}) = 1, \tag{15}$$

Problem 5. In class we proved that the operator which rotates a state $|\Psi\rangle$ by an angle θ is given by

$$\hat{\mathcal{R}}(\theta) = \cos\theta\hat{1} + i\sin\theta\hat{S} \tag{16}$$

where

$$\hat{S} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{17}$$

is the photon spin operator and and $\hat{1}$ is the unit 2×2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that

$$\hat{\mathcal{R}}(\theta) = \exp(i\theta\hat{S}) \tag{18}$$

hint: notice that $\hat{S}^2 = 1$ which implies that $\hat{S}^n = 1$ for any even power n and $\hat{S}^n = \hat{S}$ for any odd power n.

Problem 6. The probability that a photon in state $|\Psi\rangle$ passes through an x-polaroid is the average value of a physical observable which might be called "x-polarizedness." White down the operator, $\hat{P}_x$, corresponding to this observable. Show that it is Hermitian. Find its eigenstates and eigenvalues. Verify that the probability that a photon in state $|\Psi\rangle$ passes through the x-polaroid is $\langle\Psi|\hat{P}_x|\Psi\rangle$.