**Quantum Mechanics A**
**Fall 2010**
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**Problem Set 2**

## Problem 1

Consider a free particle in a one-dimensional (1D) infinite potential well of linear size $L$. Namely, the potential $V(x)$ is infinite for $x = 0$ and at $x = L$ and $V(x) = 0$ for $0 < x < L$.

1. Find the energy eigenstates and the corresponding eigenvalues.
2. Are the wavefunctions of part 1 simultaneous eigenstates of the momentum operator?
3. What is the average value of the momentum when the system is in any of the energy eigenstates found in part 1.

## Problem 2

Consider a free particle which is confined to move on a large circle of perimeter $L$. Think of the so-called free particle in 1D as the limit of the free particle moving on the above mentioned circle when $L$ becomes infinitely large. Namely, as the radius of the circle becomes bigger and bigger the path where the particle is moving becomes locally straight (no curvature).

1. Consider the state

$$\psi_k(x) = \frac{1}{\sqrt{L}}\, e^{ikx}, \qquad (0 \leq x < L). \tag{1}$$

   Show that this state is a normalized eigenstate of the free-particle Hamiltonian. What is the corresponding energy eigenvalue?

2. Show that the wavefunction given by Eq. 1 is an eigenstate of the momentum operator. What is the corresponding eigenvalue for the momentum operator?

3. Apply the appropriate periodic boundary condition to this wavefunction. What are the possible values of the wave-vector $k$?

## Problem 3

We want to generalize the previous two problems in two and three dimensions.

1. First consider a cubic box of volume $V = L^3$ where the potential at the sides of the box is infinite. This implies Dirichlet boundary conditions on the wavefunction at the six sides of the box, i.e., $\psi(0, y, z) = \psi(x, 0, z) = \psi(x, y, 0) = 0$ and $\psi(L, y, z) = \psi(x, L, z) = \psi(x, y, L) = 0$.

   Find the energy eigenstates and energy eigenvalues for a free particle in this box.

2. Are the energy eigenstates simultaneous eigenstates of the momentum operator?

3. Consider a free particle in box of volume $V = L^3$ but in this case apply periodic boundary conditions (P.B.C., also called Born-von Karman B.C) on the single-particle wavefunction. Namely, that $\psi(x, y, z) = \psi(x + L, y, z) = \psi(x, y + L, z) = \psi(x, y, z + L)$ for any position $\mathbf{r} = (x, y, z)$.

   Determine the normalized energy eigenstates in this case.

4. Are the energy eigenstates simultaneous eigenstates of the momentum operator?

5. In momentum space, draw a cubic lattice such that every lattice point to correspond to one particular state of the free particle. What is the density of states in momentum space?

6. Consider the above free-particle in the box with P.B.C. and the sum

$$\sum_{\mathbf{p}} f(\mathbf{p}) = \sum_{\mathbf{p}_x} \sum_{\mathbf{p}_y} \sum_{\mathbf{p}_y} f(\mathbf{p}) \tag{2}$$

   which is over all possible eigenvalues of the momentum operator. Show that in the limit of $L \to \infty$ this can be written as follows

$$\frac{1}{V} \sum_{\mathbf{p}} f(\mathbf{p}) \to \int \frac{d^3p}{(2\pi\hbar)^3} f(\mathbf{p}) \tag{3}$$

**Problem 4**

1. What is the Fourier transform of the 3D delta function $\delta(\mathbf{r}-\mathbf{r}')$.

2. Show that

$$\langle \mathbf{p}|\mathbf{p}'\rangle = (2\pi\hbar)^3\delta(\mathbf{p}-\mathbf{p}') \tag{4}$$

   where $|\mathbf{p}\rangle$ stands for eigenstates of the momentum.

3. Consider a wave-packet which is Gaussian in momentum space. Find the wavefunction of the particle in position space.

**Problem 5**

Consider a 3D Harmonic potential

$$V(\mathbf{r}) = \frac{1}{2}Gr^2 \tag{5}$$

What is the Schrödinger equation written in momentum space? Namely, what is the equation of motion for the amplitude $< \mathbf{p}|\psi(t)\rangle$?

**Problem 6**

We define the free-particle propagator $K(\mathbf{r},t,\mathbf{r}',t')$ as the following Green's function

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m}\right)K(\mathbf{r},t,\mathbf{r}',t') = 0 \tag{6}$$

Consider the time evolution of the wavefunction $\psi(\mathbf{r},t)$ when there is a potential $V(\mathbf{r},t)$ acting on the particle. Suppose that at some initial time $t=t_0$ the wavefunction is $\psi_0(\mathbf{r})$. Show that at a later time $t$ the wavefunction is given as a solution to the following integral equation

$$\psi(\mathbf{r},t) = \int d^3r' K(\mathbf{r},t,\mathbf{r}',t')\psi_0(\mathbf{r}') + \frac{1}{i\hbar}\int_{t_0}^{t} dt' \int d^3r' K(\mathbf{r},t,\mathbf{r}',t')V(\mathbf{r}',t')\psi(\mathbf{r}',t'). \tag{7}$$