Quantum Mechanics A
Fall 2010
Instructor: Professor E. Manousakis

Problem Set 3

Problem 1 (same as Problem 6 of Set 2

We define the free-particle propagator $K(\mathbf{r}, t, \mathbf{r}', t')$ as the following Green's function

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m}\right)K(\mathbf{r}, t, \mathbf{r}', t') = 0. \quad (1)$$

Consider the time evolution of the wavefunction $\psi(\mathbf{r}, t)$ when there is a potential $V(\mathbf{r}, t)$ acting on the particle. Suppose that at some initial time $t = t_0$ the wavefunction is $\psi_0(\mathbf{r})$. Show that at a later time t the wavefunction is given as a solution to the following integral equation

$$\psi(\mathbf{r}, t) = \int d^3r' K(\mathbf{r}, t, \mathbf{r}', t_0)\psi_0(\mathbf{r}') + \frac{1}{i\hbar}\int_{t_0}^{t} dt' \int d^3r' K(\mathbf{r}, t, \mathbf{r}', t')V(\mathbf{r}', t')\psi(\mathbf{r}', t'). \quad (2)$$

Problem 2

Start from the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2_{\mathbf{r}}\psi(\mathbf{r}, t) + V(\mathbf{r})\psi(\mathbf{r}, t), \quad (3)$$

and show that this equation can be transformed in momentum space to the following equation:

$$i\hbar\frac{\partial\phi(\mathbf{p}, t)}{\partial t} = \frac{p^2}{2m}\phi(\mathbf{p}, t) + V(i\hbar\nabla_{\mathbf{p}})\phi(\mathbf{p}, t). \quad (4)$$

how is $\phi(\mathbf{p}, t)$ defined and what is its meaning?

Problem 3

Show that the wave-packet with the minimum uncertainty between the x component of the position and the x-component of the momentum, is in the form of a Gaussian, i.e, its wavefunction is given by

$$\langle\mathbf{r}|\psi\rangle = \exp\Big(i\frac{<p>x}{\hbar} - \frac{(x-<x>)^2}{4(\Delta x)^2}\Big)g(y, z) \quad (5)$$

where $g(y, z)$ is an arbitrary function of the other two components y and z.

Problem 4

1. Start from the time-dependent Schrödinger equation (with non-zero potential) to derive the following equation

$$\frac{\partial}{\partial t}\rho(\mathbf{r}, t) + \nabla \cdot \mathbf{j}(\mathbf{r}, t), \tag{6}$$

   where $\rho(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$, i.e., it is the probability density, and

$$\mathbf{j}(\mathbf{r}, t) = \frac{1}{2m}\Big(\psi^*(\mathbf{r}, t)(-i\hbar\nabla\psi(\mathbf{r}, t)) - (-i\hbar\nabla\psi^*(\mathbf{r}, t))\psi(\mathbf{r}, t)\Big). \tag{7}$$

   Notice that Eq. 6 is the continuity equation. Thus, since $\rho(\mathbf{r}, t)$ is the probability density, the quantity $\mathbf{j}(\mathbf{r}, t)$ which is defined above and satisfies the continuity equation should be identified by the probability current density. Explain why.

2. What is the form of $\mathbf{j}(\mathbf{r}, t)$ for the case where there is a magnetic field present specified by the vector potential $\mathbf{A}(\mathbf{r}, t)$?

Problem 5

Consider a charged particle of charge e traveling in the electromagnetic potentials

$$\mathbf{A}(\mathbf{r}, t) = -\nabla\lambda(\mathbf{r}, t), \tag{8}$$

$$\phi(\mathbf{r}, t) = \frac{1}{c}\frac{\partial\lambda(\mathbf{r}, t)}{\partial t}, \tag{9}$$

where $\lambda(\mathbf{r}, t)$ is an arbitrary scalar function.

1. Determine the electromagnetic field described by these potentials.

2. Show that the wavefunction of the particle in these fields is given by

$$\psi(\mathbf{r}, t) = \exp\Big(-\frac{ie}{\hbar c}\lambda(\mathbf{r}, t)\Big)\psi^{(0)}(\mathbf{r}, t), \tag{10}$$

   where $\psi^{(0)}(\mathbf{r}, t)$ is the solution to the Schrödinger equation for the case of $\lambda(\mathbf{r}, t) = 0$.

3. For the special case where the particle feels a spatially uniform but time-dependent potential $e\phi(t)$, show that

$$\psi(\mathbf{r},t) = \exp\Big(-\frac{ie}{\hbar}\int_{-\infty}^{t}\phi(t')dt'\Big)\psi^{(0)}(\mathbf{r},t). \tag{11}$$

Problem 6

Consider doing a "two-slit interference" experiment where the slits are replaced by long conducting tubes as shown below. The source S emits particles

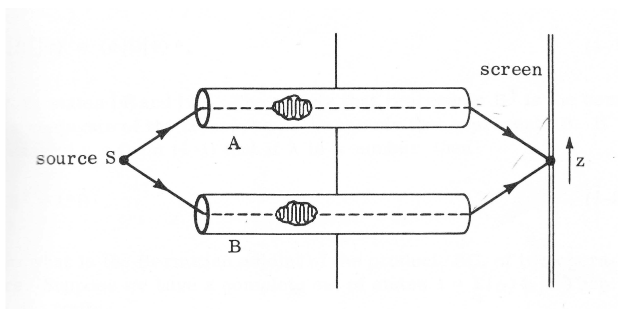

Figure 1:

in reasonably well-defined wave-packets, so that one can be sure (if the tubes are long enough) that for a certain time interval, say t_0 to t_1 seconds after emission, the wave-packet of the particle is definitely within the tubes. During this time interval, a constant voltage V_A is applied to tube A and a constant voltage V_B is applied to tube B. The rest of the time there is no voltage on the tubes. These voltages produce no fields well within the tubes.

1. Describe how the interference pattern on the screen depends on V_A and V_B.

2. If the phase difference between the amplitudes arriving at the screen from the two tubes is $k_z z$ (where $k_z = p_z/\hbar$ is the z-component of the wave-vector), and $k_z = 10^4 cm^{-1}$, by how many cm is the diffraction pattern shifted, if $V_A = 0$ and $V_B = 10^{-6}$ volts and $t_1 - t_0 = 10^{-9}$ sec?