Quantum Mechanics A
Fall 2010
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Problem Set 4

Problem 1

In class we discussed that the amplitude $K(\mathbf{r}_f, t_f; \mathbf{r}_0, t_0)$, for a particle to be at position $\mathbf{r}_f$ at time t_f given that the particle was at position $\mathbf{r}_0$ at time t_0, can be written as

$$K(\mathbf{r}_f, t_f; \mathbf{r}_0, t_0) = C \int_{\mathbf{r}(t=t_0)=\mathbf{r}_0}^{\mathbf{r}(t=t_f)=\mathbf{r}_f} \mathcal{D}\mathbf{r}(t) \exp(\frac{i}{\hbar} S[\mathbf{r}(t)]), \tag{1}$$

namely, as an integral over all paths (path-integral) $\mathbf{r}(t)$ subject to the constraint that $\mathbf{r}(t = t_0) = \mathbf{r}_0$ and $\mathbf{r}(t = t_f) = \mathbf{r}_f$. Here $S[\mathbf{r}(t)]$ is the classical action over the path $\mathbf{r}(t)$, i.e,

$$S[\mathbf{r}(t)] = \int_{t_0}^{t_f} dt L(\mathbf{r}(t), \frac{d\mathbf{r}(t)}{dt}), \tag{2}$$

and L is the corresponding classical Lagrangian of the system. The mathematical meaning of this path integral is defined as follows. Let us consider the following N independent integration variables $\mathbf{r}_1$, $\mathbf{r}_2$, ..., $\mathbf{r}_N$ which correspond to the successive time-slices t_1, t_2, ..., t_N which are introduced and split the interval (t_0, t_f) in $N + 1$ equal time intervals $t_{i+1} - t_i = \Delta t$ ($i = 0, 1, .., N$ and $t_{N+1} = t_f$). Then, the above path integral is the limit of the following N 3D integrals

$$\int_{\mathbf{r}(t=t_0)=\mathbf{r}_0}^{\mathbf{r}(t=t_f)=\mathbf{r}_f} \mathcal{D}\mathbf{r}(t) \exp(\frac{i}{\hbar} S[\mathbf{r}(t)]) \equiv \lim_{N\to\infty, \Delta t\to 0} \int d\mathbf{r}_1 d\mathbf{r}_2 ... d\mathbf{r}_N \exp(\frac{i}{\hbar} S_N[\{\mathbf{r}_i\}]), \tag{3}$$

where

$$S_N[\{\mathbf{r}_i\}] = \Delta t \sum_{i=0}^{N} L(\mathbf{r_i}(t), \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{\Delta t}), \tag{4}$$

is the discrete action on the discretized path $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$,...,$\mathbf{r}_{N+1}$, (where $\mathbf{r}_{N+1} = \mathbf{r}_f$) and the velocity variable has been also replaced by its discrete form.

When we consider the free particle case the discretized action is

$$S_N[\{\mathbf{r}_i\}] = \frac{1}{2}m\sum_{i=0}^{N}\frac{(\mathbf{r}_{i+1}-\mathbf{r}_i)^2}{\Delta t} \tag{5}$$

and the constant C in the equation 1 is given by

$$C = \Big(\frac{m}{2\pi i\hbar\Delta t}\Big)^{3(N+1)/2}. \tag{6}$$

For the free-particle case, by carrying out one-by-one the N Gaussian integrals over the variables $\mathbf{r}_1$, $\mathbf{r}_2$, ..., $\mathbf{r}_N$, show that

$$K(\mathbf{r}_f, t_f; \mathbf{r}_0, t_0) = \Big(\frac{m}{2\pi i\hbar(t_f - t_0)}\Big)^{3/2}\exp\Big(\frac{im}{2\hbar}\frac{(\mathbf{r}_f-\mathbf{r}_0)^2}{t_f - t_0}\Big). \tag{7}$$

Problem 2

Consider the 1D potential $V(x)$ illustrated in Fig. 1, such that $V(x) = 0$ for $x < 0$ and $V(x) = V$ for $x > 0$ and assume that a wave packet with

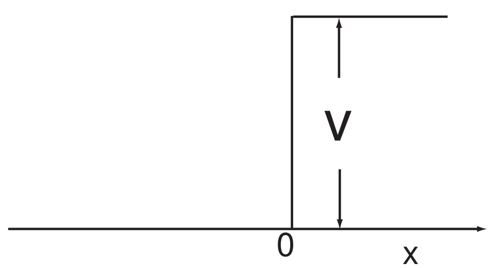

Figure 1:

energy $E_0 = p_0^2/2m < V$ is incident on the barrier from the left. Calculate in terms of E_0 and V the difference in time between the arrival of the incident packet at the step and the departure of the reflected packet from the step. Relate this time delay to a "distance of travel" within the step.

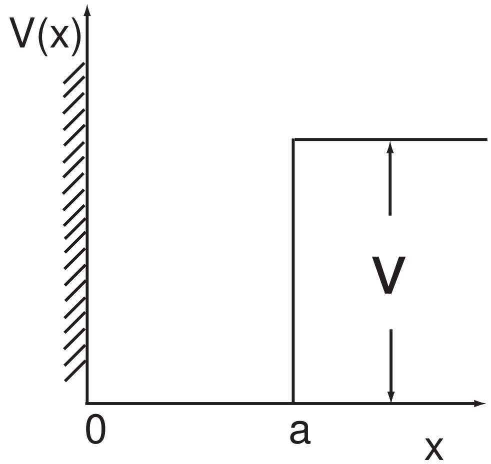

Figure 2:

Problem 3

Consider a particle in the 1D potential given by

$$V(x) = \left\{ \begin{array}{cc} \infty & x \leq 0 \\ 0 & 0 < x < a \\ V & x > a \end{array} \right\}, \tag{8}$$

which is illustrated in Fig. 2. In the limit of $V \to \infty$ the ground state wave-function of this potential is given by

$$\psi_0(x) = \sqrt{\frac{2}{a}} \sin(k_0 x) \tag{9}$$

$$k_0 = \frac{\pi}{a} \tag{10}$$

and the ground-state energy is

$$E_0 = \frac{\hbar^2 k_0^2}{2m} \tag{11}$$

a) Find the equation for the energy E of bound states in this potential.

b) Provide a schematic graphical solution for the energy eigenvalues which correspond to bound states.

c) Based on your understanding gained from the above schematic graphical solution, argue that in the limit of very large potential step V, i.e., when $E/V \rightarrow 0$, the wave-number $k = \sqrt{2mE}/\hbar$ which corresponds to the ground state is close to π/a (the case of $V \rightarrow \infty$) and more specifically,

$$ka \simeq \pi - \epsilon \tag{12}$$

and $\epsilon \rightarrow 0^+$ as $V \rightarrow \infty$. What is the leading term of ϵ in the large V limit?

d) Determine the ground-state wave-function, inside the classically forbidden region, i.e., when $E < V$, including the normalization constants, in the leading order of $\frac{1}{\lambda_0}$, where $\lambda_0 = \sqrt{2mV}a/\hbar$.

Problem 4

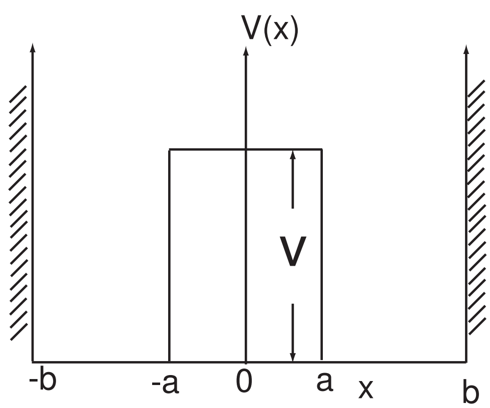

Figure 3:

Consider a particle in the 1D potential given as

$$V(x) = \left\{ \begin{array}{cc} \infty & |x| > b \\ V & |x| < a \\ 0 & a < |x| < b \end{array} \right\}, \tag{13}$$

which is illustrated in Fig. 3. We would like to study the large V limit of this potential. In the infinite V limit, the problem reduces to that of two infinite square well potentials separated by a distance $2a$. However, for a finite but very large V, (i.e., V is much larger than the energy E of the ground state), the tunneling rate between the two zero-potential sides of the well is small and the particle can be thought of as being in one of nearly two degenerate ground states $|1\rangle$ and $|2\rangle$, corresponding to being localized in the left or in the right well, respectively, each with energy $E_0 = \hbar^2 k_0^2/(2m)$ (where $k_0 = \pi/(b-a)$, in this case).

a) When V is finite but large compared to E_0, find the wave-functions of the two states $|1\rangle$ and $|2\rangle$ in the leading order in E/V, by applying the findings of the previous problem.

b) Having found the two localized states $|1\rangle$ and $|2\rangle$ in leading order in E/V, consider these two states as the only states spanning the space of your problem. In such case we can write the Hamiltonian $\hat{H}$ of your system

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x), \tag{14}$$

in the basis of these two states, as a 2×2 matrix

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \tag{15}$$

where the matrix elements h_{ij} are the matrix elements

$$h_{ij} = \int dx \psi_i^*(x) \hat{H} \psi_j(x) \tag{16}$$

between the wave-function $\psi_i(x)$ corresponding to the states $|1\rangle$ and $|2\rangle$ which you found above.

Calculate the above 2×2 matrix. Make sure that the wave-functions $\psi_1(x)$ and $\psi_2(x)$ are centered around the correct part of space within the potential well.

c) Diagonalize the above matrix and find the eigenstates and eigenvalues. What is the energy difference between the two energy eigenvalues.

d) Assume that the particle is in state $|1\rangle$ at time $t = 0$. What is the probability for a particle to tunnel from state $|1\rangle$ to state $|2\rangle$ at a later time t?