Quantum Mechanics A
Fall 2010
Instructor: Professor E. Manousakis

Problem Set 5

Problem 1

A particle of mass m is under the influence of a 1D potential of the form

$$V(x) = V\delta(x) \tag{1}$$

where $V < 0$.

a) Solve for the bound state energy and wave function.

b) Think of this potential as arising from the square well potential in the limit when it becomes very deep and narrow. Show that the bound state energy and wave-function of the delta function can be obtained in the above limit from the square-well potential.

c) While at any spatial point $x = a$ we require the wave function and its derivative to be continues, for the case of the delta function interaction, the derivative is discontinuous. This seems contradictory; explain why there is no contradiction when you consider the delta function as the above limit of the square well potential.

d) Show that the scattering amplitude $S(E)$ for the delta function potential is the limiting value of $S(E)$ for the square well potential.

Problem 2

A particle of mass m is under the influence of a 1D potential of the form

$$V(x) = V\Big(\delta(x-a) + \delta(x+a)\Big) \tag{2}$$

where $V < 0$.

a) What is the wave function for a bound state with an even parity?

b) Find the expression that determines the bound state energies for even parity states, and determine graphically how many even parity bound states there are.

c) Solve for the even parity bound state analytically in the case where $m|V|a/\hbar^2 << 1$.

d) Repeat parts (a) and (b) for odd parity. For what values of $|V|$ are there bound states?

e) Find the even and odd parity state binding energies for $a >> \hbar^2/m|V|$. Explain physically why these energies move closer and closer together as $a \to \infty$.

Problem 3

Calculate the transmission amplitude $S(E)$ for the following potential

$$V(x) = \left\{ \begin{array}{cc} 0, & 0 \leq |x| \leq \frac{b}{2} \\ V(>0), & \frac{b}{2} \leq |x| < \frac{a}{2} \\ 0, & |x| \geq \frac{a}{2} \end{array} \right\} \tag{3}$$

Calculate the resonances in $S(E)$ in the limit of large V.

Problem 4

Consider a particle in the 1D potential given as

$$V(x) = \left\{ \begin{array}{cc} 0, & |x| > a + \frac{b}{2} \\ 0, & |x| < \frac{b}{2} \\ -V, & \frac{b}{2} < |x| < a + \frac{b}{2} \end{array} \right\} \tag{4}$$

with $V > 0$, which is illustrated in Fig. 1. We would like to study the large V limit of this potential. In the infinite V limit, the problem reduces to that of two infinite square well potentials separated by a distance b. However, for a finite but very large V, (i.e., V is much larger than the energy E of the ground state), the tunneling rate between the two zero-potential sides of the well is small and the particle can be thought of as being in one of nearly two degenerate ground states $|1\rangle$ and $|2\rangle$, corresponding to being localized in the left or in the right well, respectively, each with energy $E_0 = \hbar^2 k_0^2/(2m) - V$ (where $k_0 = \pi/a$, in this case).

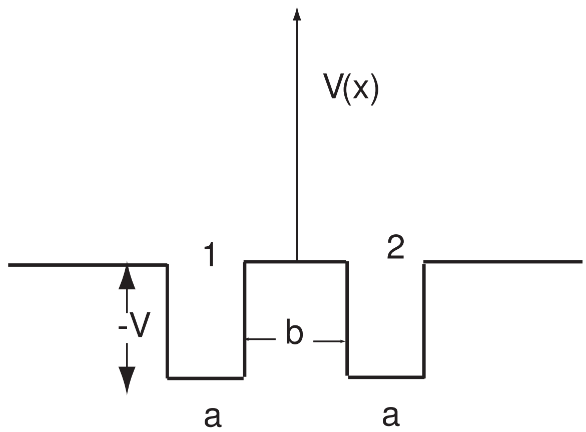

Figure 1:

a) Write down the wave-functions of the two ground states $|1\rangle$ and $|2\rangle$, which correspond to the particle being in well 1 or well 2 respectively assuming that the two wells are far away from each other. Use our calculations in class for one square well potential of well-deapth $-V$ and size a.

b) Having written down the two localized states $|1\rangle$ and $|2\rangle$, consider these two states as the only states spanning the space of your problem. In the case where the two wells are relatively close to each other neither $|1\rangle$ nor $|2\rangle$ are the eigenstates of the problem. In this case we can approximate the Hamiltonian $\hat{H}$ of our system

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x), \tag{5}$$

in the basis of these two states, as a 2×2 matrix

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \tag{6}$$

where the matrix elements h_{ij} are the matrix elements

$$h_{ij} = \int dx \psi_i^*(x) \hat{H} \psi_j(x) \tag{7}$$

between the wave-function $\psi_i(x)$ corresponding to the states $|1\rangle$ and $|2\rangle$ which you found above.

Calculate the above 2×2 matrix. Make sure that the wave-functions $\psi_1(x)$ and $\psi_2(x)$ are centered around the correct part of space within the potential well.

c) Diagonalize the above matrix and find the eigenstates and eigenvalues. What is the energy difference between the two energy eigenvalues.

d) Assume that the particle is in state $|1\rangle$ at time $t = 0$. What is the probability for a particle to tunnel from state $|1\rangle$ to state $|2\rangle$ at a later time t?