Problem Set 8

Problem 1

(a) Explicitly construct the 3×3 matrices that represent the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ in the space of $l = 1$, i.e.,

$$(\hat{L}_i)_{m,m'} = \langle l = 1, m | \hat{L}_i | l = 1, m' \rangle = \int d\Omega Y^*_{1,m}(\theta, \phi) \hat{L}_i Y_{1,m'}(\theta, \phi), \qquad (1)$$

where $i = x, y, z$.

(b) Show by explicit calculation that these three matrices obey the commutation relations of angular momentum.

(c) Find the matrices that represent the "ladder" operators, $\hat{L}_+$, $\hat{L}_-$ and also $\hat{L}^2$.

Problem 2

A quantum system is described by the following Hamiltonian

$$\hat{H} = \lambda \hat{L}_z^2 - h \hat{L}_x \qquad (2)$$

where $\hat{L}_x$ and $\hat{L}_z$ are the x and z components of the angular momentum operator. Consider the $l = 1$ subspace of the total angular momentum.

a) Find the eigenvalues of the above Hamiltonian.

b) Find the eigenstates of the Hamiltonian.

c) Imagine that we prepare the system in the initial state

$$|\psi(t = 0)\rangle = |l = 1, m = 1\rangle, \qquad (3)$$

namely, with projection of the angular momentum along the z axis characterized by $m = 1$ at time $t = 0$. Find the probability for the system to be observed in state $|l = 1, m = -1\rangle$ at time $t > 0$.

Problem 3

In class it was shown that under rotations by an infinitesimal angle α along the direction of the vector $\vec{\alpha}$, the vector $\vec{r}$ is transformed into a vector

1

$\vec{r}'$ which is given by

$$\vec{r}' = r + \vec{\alpha} \times \vec{r}, \tag{4}$$

and the state $|\vec{r}'\rangle$ is related to the state $|\vec{r}\rangle$ according to the following relation

$$\langle \vec{r}'| = \langle r + \vec{\alpha} \times \vec{r}| = \langle \vec{r}|\left(1 + \frac{i}{\hbar}\vec{\alpha} \cdot \hat{\vec{L}}\right), \tag{5}$$

where $\hat{\vec{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$, the angular momentum vector operator.

By considering rotations (a) along the z axis, (b) along the y axis and (c) along the z axis and using the above relations show that the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$, in spherical coordinates, are represented as follows

$$\hat{L}_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}), \tag{6}$$

$$\hat{L}_y = i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}), \tag{7}$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}. \tag{8}$$

Problem 4

Given the fact that the spherical harmonics $Y_{l,m}(\theta, \phi)$ form a complete basis for the angular part of any function of the spherical coordinates r, θ, ϕ expand the following functions

$$f_1 = x, \qquad f_2 = y, \qquad f_3 = z \tag{9}$$
$$f_4 = x^2 - y^2, \qquad f_5 = z^2, \qquad f_6 = xy \tag{10}$$

as a linear combination of the spherical harmonics, i.e.,

$$f_i = \phi_i(r)\sum_{l,m}c(l,m)Y_{l,m}(\theta, \phi) \tag{11}$$

where $\phi_i(r)$ are specific functions of r only. Find the coefficients $c(l,m)$.

Problem 5

(a) Find the energy levels and wave functions of a two-dimensional harmonic oscillator, with $V(r) = m\omega^2 r^2/2$ ($r^2 = x^2 + y^2$) by solving the wave

equation in Cartesian coordinates. Find the degeneracy of each level. Write out the wave functions of the ground state, and each of the first excited states.

(b) Write the Schrödinger equation for this problem in polar coordinates. Explicitly construct the wave functions of the first excited states with angular momentum $\hbar$ and $-\hbar$; these are linear combinations of the wave functions found in part (a).

Hint: Use the states which you found in part (a) to span the space of degeneracy of the first excited energy level. Show that you can make linear combinations of these states to construct eigenstates of the only angular momentum operator in 2D. Does the corresponding energy eigenvalue change when you make linear combinations of states which span the space of degeneracy? Show that these states satisfy the Schrödinger equation in polar coordinates.