Quantum Mechanics A
Fall 2010
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Problem Set 9

Problem 1

Consider a two dimensional isotropic harmonic oscillator in polar coordinates. As it was done in the Homework Set 8, the energy eigenfunctions, which are simulaneously eigenstates of the only angular momentum operator in 2D, can be written as

$$\Psi(\rho, \phi) = u(\rho) e^{iM\phi}, \tag{1}$$

where $M = 0, \pm 1, \pm 2, ...$ Show that $u(\rho)$ satisfies the following radial equation

$$(u'' + \frac{1}{\rho} u' - \frac{M^2}{\rho^2} u) + (k^2 - \lambda^2 \rho^2) u = 0 \tag{2}$$

where $k = \sqrt{2mE}/\hbar$ and $\lambda = m\omega/\hbar$. Putting

$$u = \rho^{|M|} e^{-\lambda \rho^2/2} P(\rho) \tag{3}$$

show that $P(\rho)$ satisfies the following differential equation

$$P'' + \Big(\frac{2|M|+1}{\rho} - 2\lambda\rho\Big) P' - \Big(2\lambda(|M|+1) - k^2\Big) P = 0 \tag{4}$$

Now use the transformation of variables to $t = \lambda\rho^2$ and show that P satisfies the following equation:

$$t\frac{d^2P}{dt^2} + ((|M|+1) - t)\frac{dP}{dt} - \frac{1}{2}((|M|+1) - \frac{k^2}{4\lambda})P = 0. \tag{5}$$

By writting the solution for P as a series in powers of t, i.e.,

$$P(t) = P_0 + P_1 t + P_2 t^2 + ...P_n t^n + ..., \tag{6}$$

show that in order for the solution to exist (not-diverge) the function $P(t)$ must be a polynomial of order n_r ($n_r = 0, 1, 2, ...$, an integer) and, thus, the energy of the 2D harmonic oscillator is given by

$$E = \hbar\omega(|M| + 1 + 2n_r). \tag{7}$$

Problem 2

A very elegant method for solving the hydrogen atom problem due to Schwinger, involves transforming the radial equation of the hydrogen atom into the radial equation of the two-dimensional harmonic oscillator. To carry out this procedure follow the following steps

(a) Replace the variable r by the variable $\lambda\rho^2/2$, where λ is a constant to be determined and let the radial part of the wavefunction $R_{nl}(r)$ (i.e., the full hydrogen atom wavefunction is $\psi_{nlm}(r,\theta,\phi) = R_{nl}Y_{lm}(\theta,\phi)$) be equal to $F(\rho)/\rho$. Find the differential equation satisfied by $F(\rho)$.

(b) Use to finding from the previous prblem for the Schrödinger equation for the radial part of the 2D harmonic oscillator. Show the differential equation obeyed by $F(\rho)$ satisfies the radial equation of the two-dimensional harmonic oscillator of frequency $\omega = \sqrt{-2\lambda^2 E/m}$ ($E < 0$ for bound states of the hydrogen problem) with angular momentum $|M| = 2l + 1$ and energy $2e^2\lambda$.

(c) Use the results of the previous problem and also the solution of the 2D harmonic oscillator in Cartesian coordinates to deduce the formula for the energy levels and their degeneracy of the hydrogen atom.

(d) Use this procedure to construct explicitly the normalized ground state wavefunction of the hydrogen atom.

Problem 3

For the hydrogen atom calculate all the non-zero matrix elements of x(i.e., the x-component of the vector $\vec{r}$ which gives the electron position relative to the nucleus) between the ground state and each of the $n = 2$ states.

Problem 4

Consider a particle in a 3D square well potential of finite depth, namely

$$V(r) = \left\{ \begin{array}{cc} -V_0 & r < R \\ 0 & r > R \end{array} \right\}, \tag{8}$$

(a) Give the equations which determine the eigenvalues of bound states in the above potential for angular momentum $l = 0, 1$ and 2.

(b) Solve these equations graphically for the case where $2mV_0R^2/\hbar^2 = 100$.