QMA Fall 2010

Solutions Homework Set 1

① Problem 1

1) $$\hat{M}|\psi\rangle = \begin{pmatrix} m_{xx} & m_{xy} \\ m_{yx} & m_{yy} \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} m_{xx}\psi_x + m_{xy}\psi_y \\ m_{yx}\psi_x + m_{yy}\psi_y \end{pmatrix}$$

$$\langle\phi|\hat{M}|\psi\rangle = (\phi_x^*, \phi_y^*) \begin{pmatrix} m_{xx}\psi_x + m_{xy}\psi_y \\ m_{yx}\psi_x + m_{yy}\psi_y \end{pmatrix}$$

$$= \phi_x^*(m_{xx}\psi_x + m_{xy}\psi_y) + \phi_y^*(m_{yx}\psi_x + m_{yy}\psi_y)$$

Thus

$$\langle\phi|M|\psi\rangle^* = \phi_x(m_{xx}^*\psi_x^* + m_{xy}^*\psi_y^*) + \phi_y(m_{yx}^*\psi_x^* + m_{yy}^*\psi_y^*) \quad (1)$$

Now

$$M^\dagger = \begin{pmatrix} m_{xx}^* & m_{yx}^* \\ m_{xy}^* & m_{yy}^* \end{pmatrix}$$

and

$$M^\dagger \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} = \begin{pmatrix} m_{xx}^*\phi_x + m_{yx}^*\phi_y \\ m_{xy}^*\phi_x + m_{yy}^*\phi_y \end{pmatrix}$$

(2)

$$\langle\psi|M^\dagger|\phi\rangle = (\psi_x^*, \psi_y^*) \begin{pmatrix} m_{xx}^*\phi_x + m_{yx}^*\phi_y \\ m_{xy}^*\phi_x + m_{yy}^*\phi_y \end{pmatrix}$$

$$= \psi_x^*(m_{xx}^*\phi_x + m_{yx}^*\phi_y) + \psi_y^*(m_{xy}^*\phi_x + m_{yy}^*\phi_y)$$

$$= \phi_x(\psi_x^* m_{xx}^* + \psi_y^* m_{xy}^*) + \phi_y(\psi_x^* m_{yx}^* + \psi_y^* m_{yy}^*)$$

(2)

(1)=(2)

2) First $(M|\psi\rangle)^\dagger = \langle\psi|M^\dagger$

and $(\lambda|\psi\rangle)^\dagger = \langle\psi|\lambda^*$

where by $\dagger$ in this case we mean the bra of the corresponding ket.

3) We begin from basis $|i\rangle$ $i=1,\dots,N$ and transform it to another basis $|i\rangle'$ $i=1,\dots N$ via tranformation T, i.e

$$|i\rangle' = \sum_{j=1}^{N} T_{ij} |j\rangle$$

where T_{ij} is an $N\times N$ matrix.

Then

$$'\langle i| = \sum_{j=1}^{N} \langle j| T_{ij}^{*}$$

but we need an orthonormal basis i.e

$$'\langle i|j\rangle' = \delta_{ij}$$

i.e

$$\sum_{k=1}^{N} \langle k| T_{ik}^{*} \sum_{l=1}^{N} T_{jl} |l\rangle = \delta_{ij}$$

$$\Rightarrow \sum_{k=1}^{N}\sum_{l=1}^{N} \delta_{kl} T_{ik}^{*} T_{jl} = \delta_{ij}$$

$$\sum_{k=1}^{N} T_{ik}^{*} T_{jk} = \delta_{ij}$$

$$\sum_{k=1}^{N} T_{jk} (T^\dagger)_{ki} = \delta_{ij} \Rightarrow TT^\dagger = \mathbb{1}$$

$$\Rightarrow T^\dagger = T^{-1}$$ Unitary

4) $$|\phi\rangle\langle\phi| = \begin{pmatrix}\phi_x \\ \phi_y\end{pmatrix}(\phi_x^*, \phi_y^*) = \begin{pmatrix}|\phi_x|^2 & \phi_x\phi_y^* \\ \phi_y\phi_x^* & |\phi_y|^2\end{pmatrix}$$

Now $$\left(|\phi\rangle\langle\phi|\right)^\dagger = \begin{pmatrix}|\phi_x|^2 & \phi_y^*\phi_x \\ \phi_x^*\phi_y & |\phi_y|^2\end{pmatrix} =$$

$$= |\phi\rangle\langle\phi|$$ Hermitian

5) $$\hat{S} = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \Rightarrow \hat{S}^\dagger = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} = \hat{S}$$

Problem 2:

$$|x(\theta)\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle$$

$$|y(\theta)\rangle = -\sin\theta\,|x\rangle + \cos\theta\,|y\rangle$$

$$|\psi\rangle = \psi_x|x\rangle + \psi_y|y\rangle = \langle x|\psi\rangle|x\rangle + \langle y|\psi\rangle|y\rangle$$

$$|\psi\rangle = \langle x(\theta)|\psi\rangle\,|x(\theta)\rangle + \langle y(\theta)|\psi\rangle\,|y(\theta)\rangle$$

Now

$$\langle x(\theta)|\psi\rangle = \psi_x\cos\theta + \psi_y\sin\theta \Rightarrow$$

$$i\frac{\partial}{\partial\theta}\langle x(\theta)|\psi\rangle = -i\sin\theta\,\psi_x + i\cos\theta\,\psi_y$$

Now $$\hat{S}|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} -i\psi_y \\ i\psi_x \end{pmatrix}$$

and $$\langle x(\theta)|\hat{S}|\psi\rangle = (\cos\theta, \sin\theta)\begin{pmatrix} -i\psi_y \\ i\psi_x \end{pmatrix} = -i\cos\theta\,\psi_y + i\sin\theta\,\psi_x$$

thus

$$-i\frac{\partial}{\partial\theta}\langle x(\theta)|\psi\rangle = \langle x(\theta)|\hat{S}|\psi\rangle$$

and similarly

$$-i\frac{\partial}{\partial\theta}\langle y(\theta)|\psi\rangle = \langle y(\theta)|\hat{S}|\psi\rangle$$

Problem 3

$$\mathrm{Tr}\, O = \sum_{n=1}^{N} \langle n|O|n\rangle$$

in the basis $|n\rangle$, $n=1,2,\dots N$

Now, let us consider another basis

$$|n\rangle' = \sum_{m=1}^{N} T_{nm}\, |m\rangle$$

Then, let

$$\mathrm{Tr}'\, O = \sum_{n=1}^{N} \langle n|' O\, |n\rangle' =$$

$$= \sum_{n} \sum_{m,m'} \langle m| (T^{+})_{mn}\, \hat{O}\, T_{nm'}\, |m'\rangle$$

$$= \sum_{m,m'} \langle m|\, \hat{O}\, |m'\rangle \underbrace{\sum_{n} (T^{+})_{mn}\, T_{nm'}}_{(T^{+}T)_{mm'} = \delta_{mm'}}$$

$$= \sum_{m} \langle m|\, \hat{O}\, |m\rangle$$

Problem 4:

1) $$\hat{P}_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \left(\psi_1^*, \psi_2^* \cdots \psi_N^*\right)$$

$$= \begin{pmatrix} \psi_1\psi_1^*, & \psi_1\psi_2^*, & \cdots & \psi_1\psi_N^* \\ \psi_2\psi_1^* & \psi_2\psi_2^* & \cdots & \psi_2\psi_N^* \\ \vdots & & & \\ \psi_N\psi_1^* & \psi_N\psi_2^* & \cdots & \psi_N\psi_N^* \end{pmatrix}$$

and

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1N} \\ Q_{21} & Q_{22} & \cdots & Q_{2N} \\ \vdots & & & \\ Q_{N1} & Q_{N2} & \cdots & Q_{NN} \end{pmatrix}$$

$$\mathrm{tr}(P_\psi Q) = \sum_i \left(P_\psi Q\right)_{ii} = \sum_i \sum_j (P_\psi)_{ij} Q_{ji} = \sum_i \sum_j \psi_i \psi_j^* Q_{ji} =$$

$$= \sum_{j,i} \psi_j^* Q_{ji} \psi_i = \langle\psi| Q |\psi\rangle$$

2) The expectation value of an operator Q in the mixed state is

$$\langle Q \rangle = P_1 \langle \psi_1 | Q | \psi_1 \rangle + P_2 \langle \psi_2 | Q | \psi_2 \rangle$$

Now $$\rho = P_1 |\psi_1\rangle\langle\psi_1| + P_2 |\psi_2\rangle\langle\psi_2|$$

$$T_r\, \hat{\rho} Q = P_1\, T_r\left(|\psi_1\rangle\langle\psi_1| Q\right) + P_2\, T_r\left(|\psi_2\rangle\langle\psi_2| Q\right)$$

using part 1) which we showed, i.e. that

$$T_r\left(|\psi\rangle_1 \langle\psi_1| Q\right) = \langle\psi_1| Q |\psi_1\rangle$$

and
$$T_r\left(|\psi_2\rangle\langle\psi_2| Q\right) = \langle\psi_2| Q |\psi_2\rangle$$

we find

$$T_r\, \hat{\rho}\hat{Q} = P_1 \langle\psi_1| Q |\psi_1\rangle + P_2 \langle\psi_2| Q |\psi_2\rangle$$

Thus

$$\langle Q \rangle = P_1 \langle\psi_1| \hat{Q} |\psi_1\rangle + P_2 \langle\psi_2| \hat{Q} |\psi_2\rangle$$

$$T_r \hat{\rho} = p_1 \, T_r |\psi_1\rangle\langle\psi_1| + p_2 \, T_r |\psi\rangle_2 \langle\psi_2|$$

but $T_r\left(|\psi\rangle\langle\psi|\right)$, using the representation in part 1, is given as

$$T_r\left(|\psi\rangle\langle\psi|\right) = |\psi_1|^2 + |\psi_2|^2 + \dots |\psi_N|^2 = 1$$

Thus,

$$T_r \hat{\rho} = p_1 + p_2 = 1$$

Problem 5

$$e^{i\theta\hat{S}} = 1 + i\theta\hat{S} + \frac{(i\theta\hat{S})^2}{2!} + \cdots + \frac{(i\theta\hat{S})^n}{n!} + \cdots$$

but $\hat{S}^n = \hat{1}$ if $n =$ even and

$\hat{S}^n = \hat{S}$ if $n =$ odd

then,

$$e^{i\theta S} = \left(1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots\right)\hat{1}$$

$$+ \; i\theta\hat{S} + \frac{(i\theta)^3}{3!}\hat{S} + \cdots$$

$$= \cos\theta\,\hat{1} + i\sin\theta\,\hat{S}$$

Problem 6

The observable which corresponds to $\hat{P}_x$ should satisfy

$$\langle \psi | \hat{P}_x | \psi \rangle = |\psi_x|^2$$

Therefor

$$(\psi_x^*, \psi_y^*) \begin{pmatrix} (P_x)_{xx} & (P_x)_{xy} \\ (P_x)_{yx} & (P_x)_{yy} \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} =$$

$$(\psi_x^*, \psi_y^*) \begin{pmatrix} (P_x)_{xx}\psi_x + (P_x)_{xy}\psi_y \\ (P_x)_{yx}\psi_x + (P_x)_{yy}\psi_y \end{pmatrix} =$$

$$= \psi_x^* \left((P_x)_{xx}\psi_x + (P_x)_{xy}\psi_y \right) + \psi_y^* \left((P_x)_{yx}\psi_x + (P_x)_{yy}\psi_y \right) =$$

$$= |\psi_x|^2 \quad \forall \; \psi_x, \psi_y$$

This implies that

$$(P_x)_{xx} = 1 \quad (P_x)_{xy} = 0 \quad (P_x)_{yx} = 0 \quad (P_x)_{yy} = 0$$

Therefore

$$\hat{P}_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_x^\dagger = P_x . \quad \text{(i.e Hermitian)}$$

eigenstates and eigenvalues.

$$\hat{P}_x \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \lambda \begin{pmatrix} a_x \\ a_y \end{pmatrix} \Rightarrow \left| \begin{pmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right| = 0$$

$$\lambda(1-\lambda) = 0 \Rightarrow \lambda = 0 \quad \lambda = 1$$

for $\lambda = 0$ the eigenstate is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = 0 \Rightarrow \phi_x = 0 \Rightarrow \phi_y = 1$$

and for $\lambda = 1$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = 0 \Rightarrow \phi_y = 0 \Rightarrow \phi_x = 1$$

i.e $\quad \lambda = 0 \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda = 1 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$