Quantum Mechanics
Solution
Set 9

Problem 1

The Schrödinger equation in polar coordinates for the harmonic oscillator is

\[-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) \psi(\rho, \theta) + \frac{\hbar^2}{2m \omega \rho^2} \psi(\rho, \theta) + \frac{1}{2} m \omega^2 \rho^2 \psi(\rho, \theta) = E \psi(\rho, \theta)\]

where \( \hat{L}_z = -i \hbar \frac{\partial}{\partial \theta} \)

Using

\[\psi(\rho, \theta) = u(\rho) e^{i M \theta}\]

we obtain:

\[-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) u(\rho) + \left[ \frac{\hbar^2 m^2}{2 m \omega \rho^2} + \frac{1}{2} m \omega^2 \rho^2 \right] u(\rho) = E u(\rho)\]

\[\Rightarrow \left( u''(\rho) + \frac{1}{\rho} u'(\rho) - \frac{\hbar^2}{m^2 \rho^2} u + \left( \kappa^2 - \lambda^2 \rho^2 \right) u = 0 \right.\]

with \( \kappa^2 = \frac{2 m E}{\hbar^2} \) \quad \lambda^2 = \frac{2 m \omega^2}{\hbar^2} \)
using:

\[ u(\rho) = \rho^{1|M|} e^{-\lambda \rho^2/2} \rho(\rho) \]

we find:

\[ u'(\rho) = \left( \frac{M}{\rho} - \lambda \rho \right) \rho^{1|M|} e^{-\lambda \rho^2/2} \rho(\rho) + \rho^{1|M|} e^{-\lambda \rho^2/2} \rho'/(\rho) \]

\[ u''(\rho) = \left[ \frac{1|M|(1|M|-1)}{\rho^2} - \lambda \left(2|M|+1 \right) + \lambda^2 \rho^2 \right] \rho^{1|M|} e^{-\lambda \rho^2/2} \rho(\rho) \]

\[ + \left( \frac{2|M|}{\rho} - 2 \lambda \rho \right) \rho^{1|M|} e^{-\lambda \rho^2/2} \rho(\rho) + \rho^{1|M|} e^{-\lambda \rho^2/2} \rho''/(\rho) \]

By substitution in the previous equation and after dividing by \( \rho^{1|M|} e^{-\lambda \rho^2/2} \) both sides we obtain

\[ \rho''/(\rho) + \left[ \frac{2|M|+1}{\rho} - 2 \lambda \rho \right] \rho'(\rho) - \left[ 2 \lambda (1|M|+1 \right) - k^2] \rho(\rho) = 0 \]

we now change variables to \( t = \lambda \rho^2 \)

\[ \frac{d\rho(\rho)}{d\rho} = \frac{dt}{d\rho} \frac{d\rho}{dt} = 2 \lambda \rho \frac{d\rho}{dt} \quad \Rightarrow \quad \frac{d^2\rho}{d\rho^2} = 2 \lambda \frac{d\rho}{dt} + \left(2 \lambda \rho \right) \frac{d^2\rho}{dt^2} \]

and, therefore, we obtain:

\[ 2a \frac{d\rho}{dt} + 4 + \lambda \frac{d^2\rho}{dt^2} + \frac{8\lambda}{2|M|+1} - 2 t \left[ 2 \lambda (1|M|+1 \right) - k^2] \frac{d\rho}{dt} - \left[ 2 \lambda (1|M|+1 \right) - k^2] \rho = 0 \]
\[ \Rightarrow \quad t \frac{d^2 p}{dt^2} + \left[ (1 M + 1) - t \right] \frac{dp}{dt} - \frac{1}{2} \left[ I_M + 1 - \frac{k^2}{2A} \right] p = 0 \]

We now write

\[ P = \sum_{n=0}^{\infty} p_n t^n \]

and substitute in the above equation. The coefficient of the \( t^n \) is

\[ (n+1)(n+1+1)p_{n+1} + (n+1)(1M+1)p_n - n p_n - \frac{1}{2} \left[ I_M + 1 - \frac{k^2}{2A} \right] p_n = 0 \]

and in order to satisfy the differential equation it must be set to zero.

This yields

\[ \frac{p_{n+1}}{p_n} = \frac{n + \frac{1}{2} \left[ I_M + 1 - \frac{k^2}{2A} \right]}{(n+1)(n+1+1)} \]

Assuming that \( p_n = 0 \) for all \( n \) and in the limit of large \( n \) we find that

\[ \lim_{n \to \infty} \frac{p_{n+1}}{p_n} = \frac{4}{n} \]
This implies that for large $n$

$$P_m \sim \frac{1}{n!}$$

and so

$$P(t) \sim \sum \frac{1}{n!} t^n \sim e^t$$

Thus, the solution would diverge. Therefore, in order to obtain a solution which is physically acceptable we need to require that for any value of the ratio $\frac{k^2}{2\lambda}$ and of $|M|$ there is a value of $n = n_r$ such that $P_{n_r+1} = 0$ i.e.,

$$n_r + \frac{1}{2} \left[ |M| + 1 - \frac{k^2}{2\lambda} \right] = 0$$

substituting the values of $k^2$ and $\lambda$ we obtain

$$E = \hbar \omega \left( 2n_r + |M| + 1 \right), \quad n_r = 0, 1, 2, \ldots$$

Now, let us reconcile this result with that obtained using Cartesian coordinates:

$$E = \hbar \omega \left( n_x + n_y + 1 \right)$$
so we must have:

\[ 2n_x + |M| = \eta_x + \eta_y \]

\[ \Rightarrow |M| = \eta_x + \eta_y - 2n_x \]

Examples:

\[ \eta_x + \eta_y = 1 \quad (\text{two states: } \eta_x = 1, \eta_y = 0, \eta_x = 0, \eta_y = 1) \]

since \( |M| \geq 0 \) the only possible value of \( n_x \) is 0.

This means that \( |M| = 1 \Rightarrow M = \pm 1 \) (also two states)

In general for two possibilities,

i) \( \eta_x + \eta_y = 2n_0 \quad \Rightarrow \quad n_x = 0, 1, \ldots, n_0 \)

and \( |M| = 2n_0, 2(n_0-1), \ldots, 2, 0 \)

respectively i.e.

\[ M = \pm 2n_0, \pm 2(n_0-1), \ldots, \pm 2, 0 \]

\[ \text{i.e. } 2n_0 + 1 \text{ states} \]

ii) \( \eta_x + \eta_y = 2n_0 + 1 \Rightarrow \eta_y = 0, 1, \ldots, n_0 \)

and \( |M| = 2n_0 + 1, 2(n_0-1) + 1, \ldots, 3, 1 \)

and so \( 2n_0 + 2 \) states.
Problem 2

The radial part of the Schrödinger equation is

\[
- \frac{\hbar^2}{2m} \frac{d^2}{dr^2} (r \Psi) + \left( \frac{\hbar^2 \ell (\ell + 1)}{2mr^2} - \frac{e^2}{r} - E \right) r \Psi = 0 \quad (2.1)
\]

using the transformation \( r = \lambda \rho^{1/2} \) and \( \rho = \frac{F(\rho)}{\rho} \)
we obtain:

\[
rr = \frac{3}{2} \rho F(\rho)
\]

\[
\frac{d}{dr} = \frac{1}{\lambda \rho} \frac{d}{d\rho}, \quad \frac{d}{dr^2} = \frac{1}{\lambda^2 \rho^2} \frac{d}{d\rho} \left( \frac{1}{\rho} \frac{d}{d\rho} \right)
\]

\[
\Rightarrow \frac{d^2(rr)}{dr^2} = \frac{1}{\lambda^2 \rho^2} \frac{d}{d\rho} \left( \frac{1}{\rho} \frac{d}{d\rho} \left( \frac{3}{2} \rho F(\rho) \right) \right) =
\]

\[
= \frac{1}{2\lambda^2 \rho^2} \frac{d}{d\rho} \left( \frac{1}{\rho} F(\rho) + \frac{d}{d\rho} F(\rho) \right) = \frac{1}{2\lambda^2 \rho^2} \left[ - \frac{F(\rho)}{\rho^2} + \frac{F'(\rho)}{\rho} + F''(\rho) \right]
\]

Here substitution and by multiplying both sides by \( 2\lambda^2 \rho \) yields:

\[
- \frac{\hbar^2}{2m} \left[ \frac{d^2 F}{d\rho^2} + \frac{1}{\rho} \frac{dF}{d\rho} \right] + \frac{\hbar^2 (2\ell + 1)}{2m \lambda^2 \rho^2} F = E \lambda^2 \rho^2 F - 2\lambda e F \quad (2.2)
\]
Since $E < 0$ we can define

$$\frac{1}{2} M \omega^2 = -E \lambda^2 \Rightarrow \omega = \sqrt{-\frac{2E \lambda^2}{M}} \quad (2.3)$$

and with the additional definition $\Theta = 2\lambda e^2$

we find

$$-\frac{L^2}{2M} \left[ \frac{d^2 F}{d\rho^2} + \frac{1}{\rho} \frac{dF}{d\rho} \right] + \frac{\hbar^2 \lambda^2}{2m \rho^2} F + \frac{1}{2} M \omega^2 \rho^2 F = EF \quad (2.4)$$

with $M = 2l+1 \quad (2.5)$

This equation is the same as the radial equation for the harmonic oscillator in 2D (previous problem).

Therefore we have

$$\Theta = \hbar \omega (2n_r + |M| + 1) \quad (2.6)$$

and by substitution we obtain:

$$2 \lambda e^2 = \hbar \sqrt{-\frac{2E \lambda^2}{M}} (2n_r + 2l+2)$$

$$\Rightarrow \lambda e^2 = \hbar \sqrt{-\frac{2E \lambda^2}{M}} (n_r + l+1)$$
by solving this equation for $E$ we obtain:
\[
E = -\frac{\hbar^2 \frac{m_e^4}{4\pi^2} \frac{1}{n^2}}{n^2}
\]
where $N = N_r + l + 1$ (2.7)

Given a fixed value of $N \geq 1$ then $l = 0, 1, \ldots, n-1$ and the corresponding values of $N_r$ should be $N_r = n-1, n-2, \ldots, 0$.
Thus, for a given value of $N > 1$ the total number of states (degenerate) which correspond to the same $E_n$ is
\[
\sum_{l=0}^{n-1} (2l+1) = 2 \sum_{l=0}^{n-1} l + \sum_{l=0}^{n-1} 1 = n(n-1) + n = n^2.
\]

Notice that the ground state of the hydrogen atom has $l = 0$ and $n = 1$. This means that it corresponds to $M = 1$ $N_r = 0$ harmonic oscillator states.

Namely, the radial part of the ground state of the hydrogen atom, i.e., $R_{n=1, l=0} = \frac{F(r)}{r}$ with $r = \frac{1}{2\mu} \rho^2$ can be found by finding $F(r)$, i.e., the radial part of the harmonic oscillator in 2D for $M = 1$. The case of $M = 1$ $N_r = 0$ corresponds to energy $E = 2\hbar \nu$ (from eq. 2.6). This means that it is obtained for $n_x + n_y = 1$ in cartesian coordinates, where the radial part is proportional to $F(r) \sim p e^{-\frac{\hbar \nu}{2\hbar} \rho^2}$ (notice the prefactor $p$).

Thus
\[
R_{n=1, l=0} = \frac{F(r)}{r} \sim e^{-\frac{\hbar \nu}{2\hbar} \rho^2}
\]
Therefore

\[ R_{10} = c \ e^{-\frac{m e^2}{\hbar^2} \frac{r}{\lambda}} \]

and using

\[ \frac{1}{2} p^2 = \frac{\hbar^2}{2m} \quad \text{and} \quad \omega = \sqrt{-2\frac{E_1}{m}} = \frac{\sqrt{2\lambda^2 \frac{m e^4}{\hbar^2}}}{2\hbar^2} = \frac{e^2}{\hbar} \]

we find

\[ R_{10} = c \ e^{-\frac{m e^2}{\hbar^2} \frac{r}{\lambda} \frac{1}{\lambda}} = c \ e^{-r/a_0} \]

where \( a_0 = \frac{\hbar^2}{m e^2} \).

The constant \( c \) can be determined using the normalization condition, i.e.

\[ \psi_{\text{ne}}(r, \phi, \theta) = R_{\text{ne}} Y_{\lambda m} (\phi, \theta) \]

\[ \psi_{100} (r, \phi, \theta) = R_{10} Y_{00} (\phi, \theta) = \frac{1}{\sqrt{4\pi}} \ c \ e^{-r/a_0} \]

and

\[ \int d^3r \ |\psi_{100}|^2 = 1 \Rightarrow |c|^2 \int dr \ r^2 \ e^{-2r/a_0} = 1 \]

\[ \Rightarrow |c|^2 \left( \frac{a_0}{2} \right)^3 \int_0^a r^2 \ e^{-x} \ dr = 1 \Rightarrow |c|^2 \frac{a_0^3}{4} = 1 \Rightarrow c = \frac{2}{a_0^{3/2}} \]
Problem 3

\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta; \]

\[ i) \text{ From the previous homework assignment we recall that} \]

\[ x = r \sqrt{\frac{2n}{3}} \left[ Y_{1,1,1}(\phi, \theta) - Y_{1,1,1}(\phi, \theta) \right] \]

Thus

\[ \langle n=1, \phi, \theta | x | n=2, \phi, \theta \rangle = \sqrt{\frac{2n}{3}} \int_0^\infty dr r^2 R_{10} r R_{2e} \cdot \]

\[ \int d^2 \sigma \left( Y_{1,1,1}(\phi, \theta) - Y_{1,1,1}(\phi, \theta) \right) Y_{\text{even}} (\phi, \theta) \]

Now

\[ \int d^2 \sigma \left( Y_{1,1,1}(\phi, \theta) - Y_{1,1,1}(\phi, \theta) \right) Y_{\text{even}} (\phi, \theta) = \delta_{\phi,1} \left( \delta_m, -1 - \delta_m, 1 \right) \]

Thus

\[ \langle 1, 0, 0 | x | 2, \phi, \theta \rangle = \sqrt{\frac{2n}{3}} \delta_{\phi,1} \left( \delta_m, -1 - \delta_m, 1 \right) \int_0^\infty dr r^2 R_{10} r R_{2e} \]

Now we need to compute the last integral.
\[ \int dr \ r^2 \ R_{1\alpha} \ R_{2\alpha} = \frac{2}{a_0^{3/2}} \left( 2a_0 \right)^{3/2} \frac{1}{a_0^{3/2}} \int dr \ r \ e^{-\frac{3}{2} \frac{r}{a_0}} \]

\[ = \frac{1}{\sqrt{6}} \ a_0 \int d\left( \frac{r}{a_0} \right) \left( \frac{r}{a_0} \right)^4 e^{-\frac{3}{2} \frac{r}{a_0}} \]

\[ = \frac{1}{\sqrt{6}} \ a_0 \int dx \ x^4 e^{-\frac{3}{2} x} = \frac{1}{\sqrt{6}} a_0 \left( \frac{2}{3} \right)^5 \int dy \ y^4 e^{-y} \]

\[ = \frac{4!}{\sqrt{6}} \left( \frac{2}{3} \right)^5 a_0 \]

Final result:

\[ \langle 1 \ 0 \ 0 \ | \ x \ | \ 2 \ \ell \ m \rangle = \frac{4!}{\sqrt{6}} \ \sqrt{\frac{2\pi}{3}} \left( \frac{2}{3} \right)^5 \delta_{n,1} \ (\delta_{m+1} - \delta_{m-1}) \ a_0 \]

\[ \text{ii) we also have shown in the previous homework that} \]

\[ y = i \sqrt{\frac{2\pi}{3}} \ r \left[ Y_{1,1}(\theta, \phi) + Y_{1,-1}(\theta, \phi) \right] \]

The calculation is very similar, yielding

\[ \langle 1 \ 0 \ 0 \ | \ y \ | \ 2 \ \ell \ m \rangle = i \sqrt{\frac{2\pi}{3}} \frac{4!}{\sqrt{6}} \left( \frac{2}{3} \right)^5 \delta_{n,1} \ (\delta_{m,1} + \delta_{m,-1}) \ a_0 \]
iii) for the case of $z = r \cos \theta$, we have shown that

$$z = r \sqrt{\frac{4m}{3}} Y_{l,0}(\theta, \phi)$$

Therefore we find that

$$\langle 100 | z | 12 \text{ em} \rangle = \sqrt{\frac{4m}{3}} \frac{41}{\sqrt{6}} (\frac{2}{3})^5 \delta_{e,1} \delta_{m,0} \alpha_0$$

Problem 4

In region I:

The radial part of Schrödinger equation satisfies the following equation:

$$\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2 \right) R_{m,e}(r) = 0 \quad (4.1)$$

with $k^2 = \frac{2m(V_0 - E)}{\hbar^2}$ \quad (4.2)
and the general solution is

\[ R_{\text{ne}}^I (r) = A_I J_e (kr) + B_I \eta_e (kr) \quad (4.3.a) \]

However because as \( r \to 0 \) \( R_{\text{ne}} \) should remain finite; this implies that \( B_I = 0 \).

Thus

\[ R_{\text{ne}}^I (r) = A_I J_e (kr) \quad (4.3.b) \]

In region \( II \)

\[ \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\ell (\ell + 1)}{r^2} + (i q)^2 \right] R_{\text{ne}}^I = 0 \quad (4.4.a) \]

\[ q^2 = \frac{2m|E|}{\hbar^2} \]

Therefore

\[ R_{\text{ne}}^I (r) = A_{II} h_e (i qr) + B_{II} h_e^* (i qr) \quad (4.5.a) \]

However \( \lim_{r \to \infty} h_e (i qr) \sim e^{-qr} \) and \( \lim_{r \to \infty} h_e (i qr) \sim e^{qr} \)

Thus \( B_{II} = 0 \) and this implies that

\[ R_{\text{ne}}^I (r) = A_{II} h_e (i qr) \quad (4.5.b) \]
we also have the condition \( r = R \), namely:

\[
R_{m0} \bigg|_{r=R} = R_{m0}^{\Pi} \bigg|_{r=R} \quad \text{and} \quad \frac{d}{dr} R_{m0} \bigg|_{r=R} = \frac{d}{dr} R_{m0}^{\Pi} \bigg|_{r=R}
\]

which imply

\[
A_{I} J_{L}(kR) = A_{II} h_{e}(iqr) \quad \text{and} \quad A_{I} R J_{L}(kR) = A_{II} \gamma h_{o}(iqr)
\]

and by dividing both sides of these two eqns we obtain

\[
k \frac{J_{L}'(kR)}{J_{L}(kR)} = iq \frac{h_{e}'(iqr)}{h_{e}(iqr)} \quad (4.6)
\]

where the primes refer to the derivative of the Bessel and Hankel functions \( J_{L}(x) \) and \( h_{o}(x) \) with respect to \( x \).

i) \( l = 0 \) case

\[
J_{0}(x) = \frac{\sin x}{x} \quad h_{0}(x) = -i \frac{e^{ix}}{x}
\]

\[
J_{0}'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} \quad h_{0}'(x) = \frac{e^{ix}}{x} + \frac{i}{x^2} e^{ix}
\]

Then from eq 4.6 we obtain

\[
k \frac{\cos kr - \sin kr}{\sin kr/kr} = iq \frac{1 + \frac{i}{iqr}}{1} \quad \Rightarrow
\]
\[ \frac{\tan kr}{kr} \left( \frac{\sin kr}{kr} - \frac{\sin kr}{(kr)^2} \right) = -q_r \left( 1 + \frac{1}{q_r} \right) \]

\[ \Rightarrow \quad kr \cot kr - \lambda = -q_r - \lambda \Rightarrow \]

\[ \tan kr = -\frac{kr}{q_r} \quad (4.7) \]

We now denote \( x = kr \) \( \Rightarrow \) \( x^2 = k^2 r^2 = \frac{2MV_0 k^2}{\lambda^2} - \frac{2MMEk^2}{\lambda^2} \)

\[ \Rightarrow x^2 = \lambda^2 - q_r^2 \text{ where } \lambda^2 = \frac{2MV_0 k^2}{\lambda^2} \]

and we will consider the case of \( \lambda^2 = 100 \)

Therefore with

\[ x = kr \text{ and } q_r = \sqrt{\lambda^2 - x^2} \quad (4.8) \]

Eq. 4.7 takes the form:

\[ \tan x = -\frac{x}{\sqrt{\lambda^2 - x^2}} \quad (4.9) \]

This equation can be solved graphically by plotting \( \tan x \) and \( -x/\sqrt{\lambda^2 - x^2} \) for \( \lambda^2 = 100 \).
this will be done later.

\[ l = 1 \]

\[ J_1(x) = \left( \frac{\sin x}{x} - \cos x \right) \frac{1}{x} \quad h_1(x) = \frac{1}{x} \left( -\frac{1}{x} - 1 \right) e^{ix} \]

\[ J'_1(x) = \frac{\cos x}{x^2} - 2 \frac{\sin x}{x^3} + \frac{\cos x}{x^2} + \frac{\sin x}{x} = \]

\[ = \frac{2 \cos x}{x^2} + \left( \frac{1}{x} - \frac{2}{x^3} \right) \sin x \]

\[ h'_1(x) = \left( \frac{2 i}{x^3} + \frac{1}{x^2} + \frac{1}{x^2} - \frac{i}{x} \right) e^{ix} = \]

\[ = \left( \frac{2 i}{x^3} + \frac{2}{x^2} - \frac{i}{x} \right) e^{ix} \]

using eq 4.6 we write

\[ k \left( \frac{2 \cos kr}{(kr)^2} + \left( \frac{1}{kr} - \frac{2}{(kr)^3} \right) \sin(kr) \right) \overline{\sin^{1/3} kr} \frac{(kr)^2}{kr} - \frac{\cos kr}{kr} = \frac{1}{\sqrt{q}} \frac{\left( \frac{2}{q} \frac{3}{q} - \frac{1}{q} \right)}{\frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}} \]

\[ 2kr + (kr)^2 - 2 \tan(kr) \overline{\tan(kr) - (kr)} = \frac{2 + 2(qr) + (qr)^2}{1 + (qr)} \]
\[-(1+qR) \left[ 2x + (x^2-2) \tan x \right] = (\tan x - x) \left( 2 + 2qR + (qR)^2 \right)\]

\[\Rightarrow -2x (1+qR) + x \left( 2 + 2qR + (qR)^2 \right) =\]

\[= \left[ (1+qR) (x^2-2) + (2+2qR+(qR)^2) \right] \tan x\]

\[\Rightarrow x (qR)^2 = \left\{ \frac{(1+qR) x^2 - 2 \sqrt{1+qR} + 2 (1+qR)}{x^2 + x^2 qR + (qR)^2} \right\} \tan x\]

\[\Rightarrow \tan x = \frac{x (qR)^2}{x^2 + x^2 qR + (qR)^2}\]

Using \((qR)^2 = \lambda^2 - x^2\) we find that

\[
\tan x = \frac{x \left( \lambda^2 - x^2 \right)}{x^2 \sqrt{\lambda^2 - x^2} + \lambda^2}
\]

\((4r10)\)

\((iini)\) \(\ell = 2\)

\[v_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin x - \frac{3}{x^2} \cos x\]

\[h_2(x) = \left( - \frac{3}{x^3} - \frac{3}{x^2} + \frac{i}{x} \right) e^x\]
\[
\begin{align*}
\frac{d}{dx}(x^2 - q) \tan(x) + (9 - x^2) x &= -\frac{9 + 9y + 4y^2 + y^3}{3 + 3y + y^2} \\
\tan(x) &= \frac{3x(x^2 - q + 9y + 4y^2 + y^3)}{(4x^2 - q)(3 + 3y + y^2) + (3 - x^2)(9 + 9y + 4y^2 + y^3)} \\
\tan(x) &= \frac{3x^2(1 + y) + x^2 y^2}{3x^2(1 + y) - x^2 y^3}
\end{align*}
\]

(b) we can solve equation (4.9), (4.10) and either (4.11.9) or (4.11.6) graphically. Figure shows \(\tan(x)\) with solid line. The rhs of equation (4.9), (4.10) and (4.11) are shown with dot-dash, dotted and dashed lines. The solutions are the intersections as shown.

Notice that there are 3 solutions for \(c<0\) and 3 for \(c=1\). However, there are only 2 solutions for \(c=2\).